Lecture 3: Finite Elements

Steady–state heat flow through an insulated rod

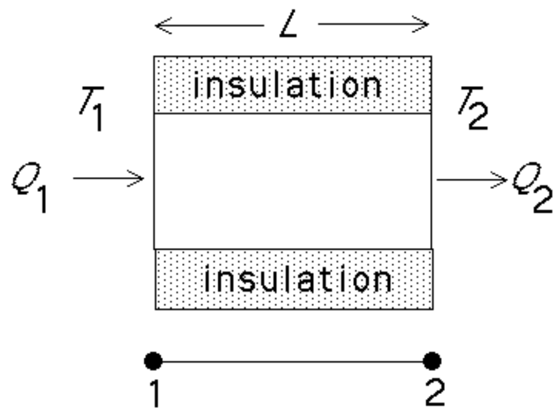

Fig. 1: One–dimensional heat flow through an insulated rod of cross–sectional area A and length L. The finite element representation consists of two nodes i and j.

Heat flow in one–dimension is described by Fourier's law, in which

$$Q = -\alpha A \frac{dT}{dx}$$

where Q is the heat flow per second through a cross–sectional area A, T is temperature, x is the coordinate along which heat flows and α is the thermal conductivity of the material in which the heat flows.

Consider heat flow through the insulated rod illustrated in Fig. 1. The heat flux entering the rod is Q_1 (defined to be positive) and that leaving the rod is Q_2. The temperatures T_1 and T_2 are maintained constant. The finite element representation consists of a single element with two nodes 1 and 2 located at x_1 and x_2 respectively. We shall assume that the temperature gradient between these nodes is uniform:

$$\frac{dT}{dx} = \frac{T_2 - T_1}{x_2 - x_1} = \frac{T_2 - T_1}{L} \qquad \text{and} \qquad Q_1 = -\alpha A \frac{T_2 - T_1}{L}$$

For steady–state heat flow,

$$Q_1 + Q_2 = 0$$

$$\text{so that} \qquad Q_2 = -\alpha A \frac{T_1 - T_2}{L}$$

These two equations can be represented in matrix form as:

$$\mathbf{Q} = \mathbf{kT}$$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \underbrace{-\frac{\alpha A}{L} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_{\mathbf{k}} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \tag{1}$$

where **k** is the thermal equivalent of the stiffness matrix.

Notice that Q_1, the heat flux entering the element, is, according to our convention, positive since $T_1 > T_2$ whereas Q_2, that leaving the element is negative.

Thermal Conduction in a Composite

Consider now the more complicated scenario illustrated in Fig. 2, consisting of a composite–rod (of unit cross–section) in which materials ‘a’, ‘b’ and ‘c’ each have different properties (Table 1).

We wish to calculate the temperatures at nodes 2 and 3, together with the heat flow per second through the rod. By inspection of equation 1, we can immediately write the matrices for elements a, b, and c as:

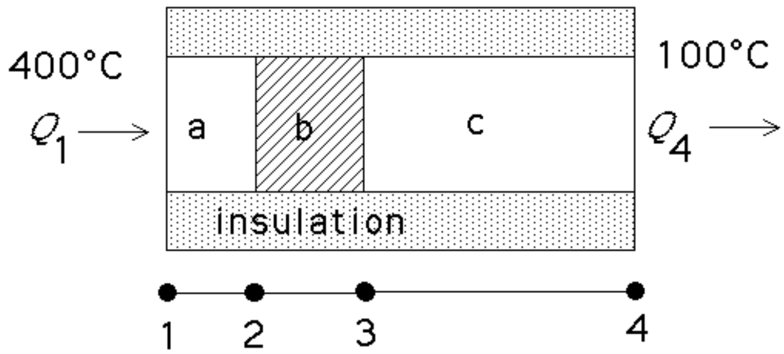

Fig. 2: One–dimensional heat flow through an insulated composite rod of unit cross–sectional area. The finite element representation consists of three elements and four nodes.

Element	Length / m	Thermal Conductivity / W m^{-1} K^{-1}
a	0.1	100
b	0.15	15
c	0.4	80

$$\mathbf{k}_a = -\frac{100}{0.1}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1000 & -1000 \\ -1000 & 1000 \end{pmatrix}$$

$$\mathbf{k}_b = -\frac{15}{0.15}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 100 & -100 \\ -100 & 100 \end{pmatrix}$$

$$\mathbf{k}_c = -\frac{80}{0.4}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 200 & -200 \\ -200 & 200 \end{pmatrix}$$

The assembled stiffness matrix thus becomes:

$$\mathbf{k} = \begin{pmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 100 & -100 & 0 \\ 0 & -100 & 100 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 200 & -200 \\ 0 & 0 & -200 & 200 \end{pmatrix}$$

$$= \begin{pmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 1000+100 & -100 & 0 \\ 0 & -100 & 100+200 & -200 \\ 0 & 0 & -200 & 200 \end{pmatrix}$$

$$\mathbf{Q} = \mathbf{kT} \qquad \text{so that}$$

$$\begin{pmatrix} Q_1 \\ 0 \\ 0 \\ Q_4 \end{pmatrix} = \begin{pmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 1100 & -100 & 0 \\ 0 & -100 & 300 & -200 \\ 0 & 0 & -200 & 200 \end{pmatrix} \begin{pmatrix} 400 \\ T_2 \\ T_3 \\ 100 \end{pmatrix}$$

Notice that $Q_2 = Q_3 = 0$ because there are no internal heat sources or sinks. It follows that $Q_1 = -Q_4 = 18750\,\mathrm{W\,m^{-2}}$, and $T_2 = 381.25\,^\circ\mathrm{C}$, $T_3 = 193.75\,^\circ\mathrm{C}$.

References

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