Appendix: Elements of Matrix Algebra

Definition, addition, scalar multiplication

A matrix is a rectangular array of numbers, having \( m \) rows and \( n \) columns, and is said to have an order \( m \) by \( n \). A square matrix \( J \) of order 3 by 3 may be written as

\[
J = \begin{pmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{pmatrix}
\]

where each number \( J_{ij} \) \((i = 1, 2, 3 \text{ and } j = 1, 2, 3)\) is an element of \( J \).

The matrix \( J' \) is called the transpose of the matrix \( J \):

\[
J' = \begin{pmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{pmatrix}
\]

An identity matrix (\( I \)) has the diagonal elements \( J_{11}, J_{22} \text{ & } J_{33} \) equal to unity, all the other elements being zero:

\[
I = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The trace of a matrix is the sum of all its diagonal elements \( J_{11} + J_{22} + J_{33} \). If matrices \( J \) and \( k \) are of the same order, they are said to be equal when \( J_{ij} = K_{ij} \) for all \( i, j \). Multiplying a matrix by a constant involves the multiplication of every element of that matrix by
that constant. Matrices of the same order may be added or subtracted, so that if \( l = J + k \), it follows that \( L_{ij} = J_{ij} + K_{ij} \).

**Multiplication and Inversion**

The matrices \( J \) and \( K \) can be multiplied in that order to give a third matrix \( L \) if the number of columns (\( m \)) of \( J \) equals the number of rows of \( K \) (\( J \) is said to be conformable to \( K \)). \( L \) is given by

\[
L_{st} = J_{sr}K_{rt}
\]

where \( s \) ranges from 1 to the total number of rows in \( J \) and \( t \) ranges from 1 to the total number of columns in \( K \). If \( J \) and \( K \) are both of order \( 3 \times 3 \) then, for example,

\[
L_{11} = J_{11}K_{11} + J_{12}K_{21} + J_{13} + K_{31}
\]

Note that the product \( JK \) does not in general equal \( KJ \).

Considering a \( n \times n \) square matrix \( J \), it is possible to define a number \( \Delta \) which is the determinant (of order \( n \)) of \( J \). A minor of any element \( J_{ij} \) is obtained by forming a new determinant of order \( (n - 1) \), of the matrix obtained by removing all the elements in the \( i \)th row and the \( j \)th column of \( J \). For example, if \( J \) is a \( 2 \times 2 \) matrix, the minor of \( J_{11} \) is simply \( J_{22} \). If \( J \) is a \( 3 \times 3 \) matrix, the minor of \( J_{11} \) is:

\[
\begin{vmatrix}
J_{22} & J_{23} \\
J_{32} & J_{33}
\end{vmatrix} = J_{22}J_{33} - J_{23}J_{32}
\]

where the vertical lines imply a determinant. The cofactor \( j_{ij} \) of the element \( J_{ij} \) is then given by multiplying the minor of \( J_{ij} \) by \((-1)^{i+j}\). The determinant (\( \Delta \)) of \( J \) is thus

\[
\det J = \sum_{j=1}^{n} J_{1j} j_{1j} \quad \text{with } J = 1, 2, 3
\]
Hence, when \( J \) is a \( 3 \times 3 \) matrix, its determinant \( \Delta \) is given by:

\[
\Delta = J_{11}j_{11} + J_{12}j_{12} + J_{13}j_{13} \\
= J_{11}(J_{22}J_{33} - J_{23}J_{32}) \\
+ J_{12}(J_{23}J_{31} - J_{21}J_{33}) \\
+ J_{13}(J_{21}J_{32} - J_{22}J_{31})
\]

The inverse of \( J \) is written \( J^{-1} \) and is defined such that

\[
J.J^{-1} = I
\]

The elements of \( J^{-1} \) are \( J^{-1}_{ij} \) such that:

\[
J^{-1}_{ij} = j_{ji}/\det J
\]

Hence, if \( L \) is the inverse of \( J \), and if \( \det J = \Delta \), then:

\[
L_{11} = (J_{22}J_{33} - J_{23}J_{32})/\Delta \\
L_{12} = (J_{32}J_{13} - J_{33}J_{12})/\Delta \\
L_{13} = (J_{12}J_{23} - J_{13}J_{22})/\Delta \\
L_{21} = (J_{23}J_{31} - J_{21}J_{33})/\Delta \\
L_{22} = (J_{33}J_{11} - J_{31}J_{13})/\Delta \\
L_{23} = (J_{13}J_{21} - J_{11}J_{23})/\Delta \\
L_{31} = (J_{21}J_{32} - J_{22}J_{31})/\Delta \\
L_{32} = (J_{31}J_{12} - J_{32}J_{11})/\Delta \\
L_{33} = (J_{11}J_{22} - J_{12}J_{21})/\Delta
\]
Example 1

\[ A = \begin{pmatrix} 2 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{pmatrix} \quad A' = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 5 & 7 \end{pmatrix} \]

\[ \det A = -8 \quad A^{-1} = \begin{pmatrix} 0.25 & -1.5 & 1 \\ -0.5 & -0.5 & 0.5 \\ 0.25 & 1.5 & -1 \end{pmatrix} \]

Example 2

\[ A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad A' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \]

\[ \det A = 5 \quad A^{-1} = \begin{pmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{pmatrix} \]