Appendix: Elements of Matrix Algebra

Definition, addition, scalar multiplication

A matrix is a rectangular array of numbers, having m rows and n columns, and is said to have an order m by n. A square matrix **J** of order 3 by 3 may be written as

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}$$

where each number J_{ij} ($i = 1, 2, 3$ and $j = 1, 2, 3$) is an element of **J**. The matrix **J'** is called the transpose of the matrix **J**:

$$\mathbf{J}' = \begin{pmatrix} J_{11} & J_{21} & J_{31} \\ J_{12} & J_{22} & J_{32} \\ J_{13} & J_{23} & J_{33} \end{pmatrix}$$

An identity matrix (**I**) has the diagonal elements J_{11}, J_{22} & J_{33} equal to unity, all the other elements being zero:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The trace of a matrix is the sum of all its diagonal elements $J_{11} + J_{22} + J_{33}$. If matrices **J** and **k** are of the same order, they are said to be equal when $J_{ij} = K_{ij}$ for all i, j. Multiplying a matrix by a constant involves the multiplication of every element of that matrix by that constant. Matrices of the same order may be added or subtracted, so that if $\mathbf{l} = \mathbf{J} + \mathbf{k}$, it follows that $L_{ij} = J_{ij} + K_{ij}$.

Multiplication and Inversion

The matrices **J** and **K** can be multiplied in that order to give a third matrix **L** if the number of columns (m) of **J** equals the number of rows of **K** (**J** is said to be conformable to **K**). **L** is given by

$$L_{st} = J_{sr} K_{rt}$$

where s ranges from 1 to the total number of rows in **J** and t ranges from 1 to the total number of columns in **K**. If **J** and **K** are both of order 3×3 then, for example,

$$L_{11} = J_{11} K_{11} + J_{12} K_{21} + J_{13} + K_{31}$$

Note that the product **JK** does not in general equal **KJ**.

Considering a $n \times n$ square matrix **J**, it is possible to define a number Δ which is the determinant (of order n) of **J**. A *minor* of any element J_{ij} is obtained by forming a new determinant of order $(n-1)$, of the matrix obtained by removing all the elements in the ith row and the jth column of **J**. For example, if **J** is a 2×2 matrix, the minor of J_{11} is simply J_{22}. If **J** is a 3×3 matrix, the minor of J_{11} is:

$$\begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} = J_{22} J_{33} - J_{23} J_{32}$$

where the vertical lines imply a determinant. The cofactor j_{ij} of the element J_{ij} is then given by multiplying the minor of J_{ij} by $(-1)^{i+j}$. The determinant (Δ) of **J** is thus

$$\det \mathbf{J} = \sum_{j=1}^{n} J_{1j} j_{1j} \qquad \text{with} J = 1, 2, 3$$

Hence, when $\mathbf{J}$ is a 3×3 matrix, its determinant Δ is given by:

$$\begin{aligned}
\Delta &= J_{11}j_{11} + J_{12}j_{12} + J_{13}j_{13} \\
&= J_{11}(J_{22}J_{33} - J_{23}J_{32}) \\
&\quad + J_{12}(J_{23}J_{31} - J_{21}J_{33}) \\
&\quad + J_{13}(J_{21}J_{32} - J_{22}J_{31})
\end{aligned}$$

The inverse of $\mathbf{J}$ is written $\mathbf{J}^{-1}$ and is defined such that

$$\mathbf{J}.\mathbf{J}^{-1} = \mathbf{I}$$

The elements of $\mathbf{J}^{-1}$ are J_{ij}^{-1} such that:

$$J_{ij}^{-1} = j_{ji}/\det\mathbf{J}$$

Hence, if $\mathbf{L}$ is the inverse of $\mathbf{J}$, and if $\det\mathbf{J} = \Delta$, then:

$$\begin{aligned}
L_{11} &= (J_{22}J_{33} - J_{23}J_{32})/\Delta \\
L_{12} &= (J_{32}J_{13} - J_{33}J_{12})/\Delta \\
L_{13} &= (J_{12}J_{23} - J_{13}J_{22})/\Delta \\
L_{21} &= (J_{23}J_{31} - J_{21}J_{33})/\Delta \\
L_{22} &= (J_{33}J_{11} - J_{31}J_{13})/\Delta \\
L_{23} &= (J_{13}J_{21} - J_{11}J_{23})/\Delta \\
L_{31} &= (J_{21}J_{32} - J_{22}J_{31})/\Delta \\
L_{32} &= (J_{31}J_{12} - J_{32}J_{11})/\Delta \\
L_{33} &= (J_{11}J_{22} - J_{12}J_{21})/\Delta
\end{aligned}$$

Example 1

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{pmatrix} \qquad \mathbf{A}' = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 5 & 7 \end{pmatrix}$$

$$det\ \mathbf{A} = -8 \qquad \mathbf{A}^{-1} = \begin{pmatrix} 0.25 & -1.5 & 1 \\ -0.5 & -0.5 & 0.5 \\ 0.25 & 1.5 & -1 \end{pmatrix}$$

Example 2

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \qquad \mathbf{A}' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$det\ \mathbf{A} = 5 \qquad \mathbf{A}^{-1} = \begin{pmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{pmatrix}$$